# Clique partitioning of interval graphs with submodular costs on the cliques

Dion Gijswijt*, Vincent Jost, Maurice Queyranne†

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**Abstract**

Given a graph $G = (V, E)$ and a "cost function" $f : 2^V \rightarrow \mathbb{R}$ (provided by an oracle), the problem [PCliqW] consists in finding a partition into cliques of $V(G)$ of minimum cost. Here, the cost of a partition is the sum of the costs of the cliques in the partition. We provide a polynomial time dynamic program for the case where $G$ is an interval graph and $f$ belongs to a subclass of submodular set functions, which we call "value-polymatroidal". This provides a common solution for various generalizations of the coloring problem in co-interval graphs such as max-coloring, "Greene-Kleitman's dual", probabilist coloring and chromatic entropy. In the last two cases, this is the first polytime algorithm for co-interval graphs. In contrast, NP-hardness of related problems is discussed. We also describe an ILP formulation for [PCliqW] which gives a common polyhedral framework to express min-max relations such as $\overline{\chi} = \alpha$ for perfect graphs and the polymatroid intersection theorem. This approach allows to provide a min-max formula for [PCliqW] if $G$ is the line-graph of a bipartite graph and $f$ is submodular. However, this approach fails to provide a min-max relation for [PCliqW] if $G$ is an interval graphs and $f$ is value-polymatroidal.

**Keywords:** Partition into cliques; Interval graphs; Circular arc graphs; Max-coloring; Probabilist coloring; Chromatic entropy; Partial $q$-coloring; Batch-scheduling; Submodular functions; Bipartite matchings; Split graphs.

*Dep. of Operations Research, EGRES, Eötvös Lorand University, Pázmány Peter Setany. 1/C, 1117 Budapest, Hungary, dion.gijswijt@gmail.com

†CNRS, laboratoire Leibniz-IMAG, 46 avenue Félix Viallet, 38031 Grenoble Cedex, France {vincent.jost, maurice.queyranne}@imag.fr

# 1 Introduction

Let $G = (V, E)$ be a simple graph. In the following, a ***clique*** of $G$ refers to a non-empty subset of vertices inducing a complete subgraph (not necessarily maximal with this property). Let $\mathcal{C}(G)$ denote the set of cliques of $G$. A partition into cliques of $G$ is a partition $\mathcal{Q} = (K_1, \ldots, K_k)$ of $V(G)$, where $K_1, \ldots, K_k \in \mathcal{C}(G)$. In other words it is a coloring of $\overline{G}$, the complementary graph of $G$. Let $\mathcal{P}(G)$ denote the set of all partitions into cliques of $G$. A classical problem consists in determining $\overline{\chi}(G)$, the minimum number of cliques necessary to partition $G$. In several applications however (see section 3), there is a ***cost*** $f(C)$ associated to every clique $C \in \mathcal{C}(G)$, and we are interested in partitioning $G$ into cliques, minimizing the sum of the costs of the cliques in the partition. Let $\overline{\chi}(G, f)$ denote this minimum:

$$(1) \qquad \overline{\chi}(G, f) := \min_{\mathcal{Q} \in \mathcal{P}(G)} \sum_{K \in \mathcal{Q}} f(K).$$

In order to describe some properties of $f$, one may assume that $f$ is not only defined on cliques but is a ***set function on*** **V**, that is $f : 2^V \to \mathbb{R}$. This has no consequences for the definitions of $\overline{\chi}(G, f)$ and [PCliqW] below. Notice that if $f(C) = 1$ for all cliques $C$, we get the classical problem of coloring $\overline{G}$ and we have $\overline{\chi}(G, \mathbf{1}) = \overline{\chi}(G)$. Determining $\overline{\chi}(G, f)$ is therefore an NP-hard problem. Moreover, since $|\mathcal{C}(G)|$ is usually exponential in $|V|$ (the complete graph $K_n$ on $n$ vertices has $|\mathcal{C}(K_n)| = 2^n$), encoding $f$ itself raises complexity issues. In several applications however, both $G$ and $f$ have structural properties that allow to solve problem [PCliqW] in time polynomial in $|V|$.

**[PCliqW] Partition into cliques with weights**
**INPUT :** A graph $G = (V, E)$ and a value oracle, providing $f(K)$ in constant time for each $K \in \mathcal{C}(G)$.
**OUTPUT :** A partition into cliques of cost $\overline{\chi}(G, f)$.

[PCliqW] can also be described in terms of batch scheduling with compatibility graphs [12]. In this terminology (see [4] for batch scheduling problems not involving compatibility graphs and [16] for a classification of chromatic scheduling problems), each clique of a partition into cliques of $G$ is called a ***batch***. The operating time of a batch $K$ is then $f(K)$ and our objective is to minimize the makespan $\mathrm{C}_{\max}$ (whence the batches are ordered arbitrarily on the batch machine). Talking about cliques and batches allows to distinguish easily between cliques of $G$ and cliques in a partition of $V(G)$. Two famous polytime cases of [PCliqW] are when

- $G$ is perfect and $f \equiv 1$ [17],
- $G$ is complete and $f$ is submodular set function [17]

Our solution for [PCliqW] for interval graphs and value-polymatroidal functions can be seen as a compromise between these two classical cases. Moreover, [PCliqW] enjoys a simple min-max formula in both cases [17] ($\overline{\chi}(G) = \alpha(G)$ in the first case and "Dilworth's truncation" in the second). One could therefore expect a common generalized min-max formula to hold in other cases for which [PCliqW] is polynomial. We deal with this issue in section 7.

In section 2, we define polymatroid rank functions and motivate the definition of value-polymatroidal set functions in the context of [PCliqW]. In section 3, we provide examples of value-polymatroidal set functions. In section 4, we discuss value-polymatroidal functions whose values $f(U)$ depend only on the size $|U|$. In section 5, we provide a dynamic program which solves [PCliqW] for interval graphs in polytime if $f$ is value-polymatroidal. The algorithm extends to the minimum cost partition problem for circular arc graphs, when we only consider cliques in which the arcs share a common point. As a counterpart, we mention NP-hardness of [PCliqW] for interval graphs if $f$ is only assumed to be polymatroidal [2]. In section 6, we discuss NP-hardness of [PCliqW] on split graphs for subclasses of value-polymatroidal set functions. In section 7, we deal with some polyhedral issues and provide a min-max formula for [PCliqW] in line-graphs of bipartite graphs.

# 2 Value-polymatroidal set functions

A set function $f : \mathcal{P}(V) \to \mathbb{R}$ is ***submodular*** if it satisfies one of the following equivalent properties [17]:

(2) $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$ for all $S, T \subseteq V$,

(3) $f(S + u) + f(T) \leq f(S) + f(T + u)$ for all $T \subseteq S \subseteq V$ and $u \in V \backslash S$,

(4) $f(S + u + v) + f(S) \leq f(S + u) + f(S + v)$ for all $S \subseteq V$ and $u, v \in V \backslash S$.

A set function $f$ is *non-negative* if all its values are, *non-decreasing* if $S \subseteq T \Longrightarrow f(S) \leq f(T)$, *subcardinal* if $f(U) \leq |U|$ for all $U \subseteq V$. A ***polymatroid*** rank function is a submodular, non-negative, non-decreasing set function such that $f(\emptyset) = 0$. A *matroid* rank function is a subcardinal, integral polymatroid rank function.

In some graph classes, submodularity of $f$ is enough to ensure polynomiality of [PCliqW] (see section 7 and [16]). Although submodularity is not sufficient for interval graphs (see Theorem 5.5), a stronger exchange property will do. We say that $f$ is a ***value-polymatroidal*** set function if $f(\emptyset) = 0$, $f$ is non-decreasing and for every $S$ and $T$ subsets of $V$ such that $f(S) \geq f(T)$ and every $u \in V \backslash (T \cup S)$, we have

(5) $f(S + u) + f(T) \leq f(S) + f(T + u)$.

**Proposition 2.1** *Every value-polymatroidal set function is a polymatroid rank function.*

**Proof** Let $f$ be value-polymatroidal. Since $f$ is non-decreasing, we have $f(S) \geq f(T)$ for every $T \subseteq S \subseteq V$ and therefore $f(S + u) + f(T) \leq f(S) + f(T + u)$ for every $u \in V \backslash S$. $\square$

By a ***maximal clique***, we mean a clique maximal for inclusion (not necessarily for cardinality). The main motivation behind the definition of value-polymatroidal set functions is given by the following proposition.

**Proposition 2.2** *For any graph $G$ and any value-polymatroidal set function $f$ on $V(G)$, there is a partition $\mathcal{Q}$ of cost $\overline{\chi}(G,f)$ in which one of the cliques in $\mathcal{Q}$ is a maximal clique of $G$.*

**Proof** Let $\mathcal{Q}$ be a minimum cost partition of $G$ and choose any clique $K \in \mathcal{Q}$, such that $f(K) \geq f(T)$ for all $T \in \mathcal{Q}$. If $K$ is not a maximal clique of $G$, there exists some $t \in V \backslash K$ such that $K+t$ is a clique in $G$. Now, $t$ belongs to some $T \in \mathcal{Q} - K$. Since $f$ is non-decreasing, $f(K) \geq f(T) \geq f(T-t)$. Since $f$ is value-polymatroidal, $f(K+t)+f(T-t) \leq f(K)+f(T)$. Repeat the process until $K$ becomes a maximal clique of $G$. □

In general, rank functions of (poly)matroids are not value-polymatroidal, and the conclusion of Proposition 2.2 doesn't hold as shown in Figure 1.

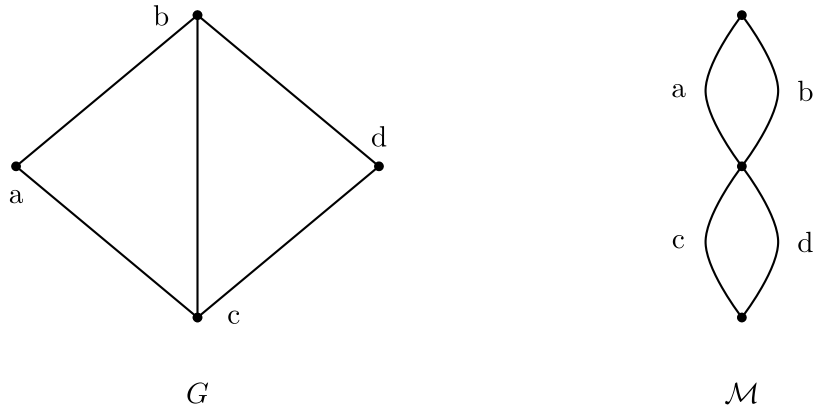


Figure 1: A graph $G$ and a graphic matroid $\mathcal{M}$ (whose rank function is not value-polymatroidal) such that $\overline{\chi}(G, r(\mathcal{M})) = 2 = r(\{a,b\}) + r(\{c,d\})$. No optimal partition contains a maximal clique of $G$.

# 3 Examples of value-polymatroidal set functions

In this section we mention some (coloring) problems that have been studied in the literature, and that amount to solving [PCliqW] for special subclasses of value-polymatroidal set functions. These problems are often formulated is terms of finding a minimum cost partition into stable sets, which is equivalent to [PCliqW] by taking the complementary graph.

**Maximum** Let $p : V \to \mathbb{R}_+$ and define

(6) $f(U) := \max_{u \in U} p(u)$

for any $U \subseteq V$. Then $f$ is value-polymatroidal. Indeed, let $S, T \subseteq V$ with $f(S) \geq f(T)$, and let $u \in V \backslash (S \cup T)$. Then, since $p(s) = f(S) \geq f(T) = p(t)$ for some $s \in S$ and $t \in T$, we have

$$f(S+u)+f(T) = \max\{p(s), p(u)\}+p(t) \leq p(s)+\max\{p(t), p(u)\} = f(S)+f(T+u).$$

A set function arising as in (6) is called a ***max-batch cost function***. When restricted to max-batch cost functions, the corresponding problem of finding a minimum cost partition into stable sets is called [max-coloring] and is strongly-NP-hard for split graphs [8, 3], for bipartite graphs [8] and for interval graphs [11]. However, [max-coloring] is polynomial for $P_4$-free graphs [8] as well as for co-interval graphs [12, 2, 9].

**Independent probabilities** Let $q : V \to [0, 1]$ and for $U \subseteq V$, let

(7) $f(U) := 1 - \Pi_{u \in U} q(u)$

Let $S, T \subseteq V$ with $f(S) \geq f(T)$, and $u \in V \backslash (S \cup T)$. Write $f(S) = 1 - \sigma$ and $f(T) = 1 - \tau$ (so $\sigma \leq \tau$). Then

$$\begin{aligned} f(S) + f(T + u) &= (1 - \sigma) + (1 - q(u)\tau) \\ &\geq (1 - q(u)\sigma) + (1 - \tau) = f(S + u) + f(T). \end{aligned}$$

Hence $f$ is value-polymatroidal. A set function arising as in (7) is a ***probabilistic cost function***. Transitive references for applications of probabilist optimization can be found in [7].

When restricted to probabilistic cost functions, [PCliqW] is strongly NP-hard in split graphs [7]. The corresponding problem of partitioning into stable sets is called [probabilist coloring].

**Chromatic Entropy** Let $p : V \to [0, 1]$ and for $U \subseteq V$, let

$$\begin{aligned} (8) \qquad c_U &:= \sum_{u \in U} p(u) \\ (9) \qquad f'(U) &:= -c_U \log(c_U). \end{aligned}$$

If $c_V = 1$, $f'$ is a ***chromatic entropy*** cost function. Although $f'$ is not value-polymatroidal (it is not non-decreasing), the function $f := f' + c$ *is* value-polymatroidal as can be derived from the concavity of the function $x \mapsto x - x \log(x)$ [1]. Since for any partition $V = K_1 \cup \cdots \cup K_k$ of $V$ into cliques, we have $\sum_i f(K_i) = c(V) + \sum_i f'(K_i)$, the two functions $f'$ and $f$ yield the same optimal partitions.

The corresponding problem of partitioning into stable sets is called [chromatic entropy] [1, 6] and is strongly NP-hard for interval graphs [6].

**Uniform matroid and Partial $q$-coloring** Let $q \in \mathbb{N}$ and let

(10) $f(U) := \min\{q, |U|\}$

Then $f$ is value-polymatroidal, and the proof is left as an exercise since a more general statement is given with the next example. Functions arising this way are exactly the rank functions of uniform matroids. [PCliqW] with such a cost function arises in Greene-Kleitman's min-max relations stating that for any (co)-comparability graph $G$ and any integer $q$, the maximum cardinality $\alpha_q(G)$ of the union of $q$ stable sets of $G$ satisfies $\alpha_q(G) = \overline{\chi}(G, f)$ (see [5] and [17], sections 14.6 and 14.7 on unions of chains and antichains in posets and section 66.5$e$ on "$k$-perfect" graphs for more details and references).

**Size-defined concave** Assume that $f(\emptyset) = 0$ and that

(11) $f(U) := \psi(|U|)$

for some $\psi : \mathbb{N} \to \mathbb{R}_+$. Then $f$ is value-polymatroidal if and only if $f$ is the rank of a polymatroid and also if and only if $\psi$ has a non-decreasing concave extension on the real segment $[0, |V|]$ (see section 4). The rank function of a uniform matroid is a special case.

# 4 Size-defined submodular set functions

In this section, we notice that if $f(U)$ only depends on $|U|$, then polymatroid ranks coincide with value-polymatroidal functions. Let $[a..b]$ denote the set of integers in the interval $[a, b]$. A set function $f$ on $V$ is ***size-defined*** if there exists a function $\psi : [0..|V|] \to \mathbb{R}$ such that $f(U) = \psi(|U|)$. The function $\psi$ is then the ***compact representation*** of $f$. Recall that a function $f : [a, b] \to \mathbb{R}$ is ***concave*** if for all $c, d \in [a, b]$ we have $f(c) + f(d) \leq 2f((c + d)/2)$

**Theorem 4.1** *Let $f$ be a size-defined, non-decreasing set function such that $f(\emptyset) = 0$ and $\psi$ be the compact representation of $f$. The following are equivalent:*

i) $f$ is value-polymatroidal

ii) $f$ is a polymatroid rank function

iii) $2\psi(i) \geq \psi(i-1) + \psi(i+1)$ for all $i \in [1..|V|-1]$

iv) $\psi(i+1) - \psi(i) \geq \psi(j+1) - \psi(j)$ for all $i, j \in [0..|V|-1]$, with $i < j$

v) $\exists\, \widehat{\psi} : [0, |V|] \to \mathbb{R}$ concave such that $\psi(i) = \widehat{\psi}(i)$ for $i \in [0..|V|]$

**Proof** i) $\Longrightarrow$ ii): Proposition 2.1
ii) $\Longrightarrow$ iii): Use definition (4) of polymatroids with $|S| = i - 1$.
iii) $\Longrightarrow$ iv): By induction on $j - i$. The case $j - i = 1$ being exactly iii). Adding $\psi(i+1) - \psi(i) \geq \psi(j+1) - \psi(j)$ and $2\psi(j+1) \geq \psi(j) + \psi(j+2)$ gives $\psi(i+1) - \psi(i) \geq$

$\psi(j+2) - \psi(j+1)$.
iv) $\Longrightarrow$ i): For $S, T \subseteq V$, since $f$ is size-defined and non-decreasing,

$$f(S) \geq f(T) \Longleftrightarrow \psi(|S|) \geq \psi(|T|) \Longleftrightarrow |S| \geq |T|$$

Applying iv) to $j = |S|$ and $i = |T|$ gives i).
v) $\Longrightarrow$ iii): Apply the concavity condition to $c = i - 1$ and $d = i + 1$.
iii) $\Longrightarrow$ v): Take $\widehat{\psi}$ as the piecewise linear interpolation of $f$ (for any $x \in [0..|V|]$, $\widehat{\psi}(x) := \lambda f(\lfloor x \rfloor) + (1 - \lambda) f(\lceil x \rceil)$ for $\lambda := x - \lfloor x \rfloor$). One can check that the subgradient of $-\widehat{\psi}$ is nondecreasing. □

# 5 Partition into cliques in interval and circular arc graphs

A graph $G = (V, E)$ is an ***interval graph*** [13, 17] if there exists a set $\{\phi(v) \mid v \in V\}$ of closed intervals on the real line, such that two vertices $u$ and $v$ are adjacent in $G$ if and only if the two corresponding intervals $\phi(u)$ and $\phi(v)$ have nonempty intersection. Observe that any maximal clique $K$ in $G$ is of the form $\{v \in V \mid t \in \phi(v)\}$ for some endpoint $t$ of one of the intervals.

In [12, 9, 2], [PCliqW] is solved in polytime for interval graphs and max-batch cost functions. These algorithms use the fact that there exists an optimal solution in which a vertex of maximum cost is contained in a batch inducing a maximal clique. Based on this fact, a dynamic program is proposed. This fact is no longer true for value-polymatroidal costs as shown by the example in Figure 2. Nonetheless, based on Lemma 5.2, we describe a generalization of the algorithm proposed in [12], which provides an optimal solution for any value-polymatroidal cost function.

**Theorem 5.1** *For any interval graph $G = (V, E)$ and any value-polymatroidal set function $f$ on $V$ given by a value oracle, we can compute a partition into cliques of $G$ of cost $\overline{\chi}(G, f)$ in time $O(n^3)$.*

**Proof** Let $\{I_i = [a_i, b_i]\}_{i=1,\ldots,n}$ be a set of intervals on the real line representing graph $G$. We consider the set $X$ of ***endpoints*** of the intervals:

$$X = \{a_i\}_{i=1,\ldots,n} \cup \{b_i\}_{i=1,\ldots,n} = \{1, \ldots, q\}.$$

Let the ***subproblem*** $\mathcal{I}(i, j)$ denote the set of all intervals completely contained in the closed interval $[i, j]$. For every pair of values $i \leq j \in X$, let $F(i, j) := \overline{\chi}(G[\mathcal{I}(i, j)], f)$, be the optimum cost of a partition of the subgraph induced by $\mathcal{I}(i, j)$ (by definition of $\overline{\chi}(G, f)$, $F(i, j) = 0$ if $\mathcal{I}(i, j) = \emptyset$). Our Dynamic Programming approach is based on Lemma 5.2 below, which implies that we can separate the problem restricted to $\mathcal{I}(i, j)$ into two subproblems.

**Lemma 5.2** *For every $i, j \in X$ there is an optimal partition into cliques of $G[\mathcal{I}(i, j)]$ in which at least one batch induces a maximal clique of $G[\mathcal{I}(i, j)]$.*

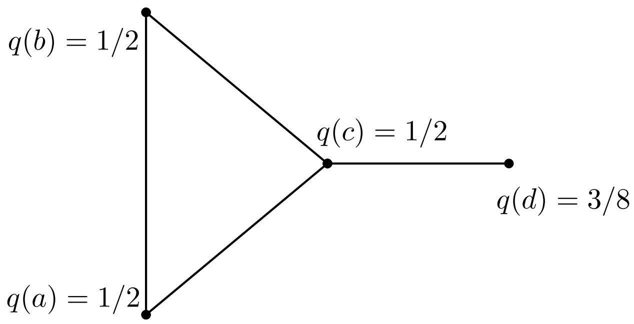


Figure 2: Let $f$ be the probabilist cost defined by $p$. Vertex $d$ has maximum cost $f(\{d\}) = 1 - q(d) = 5/8$. However, in an optimal partition, vertex $d$ cannot be placed in a maximal clique since $25/16 = f(\{a, b\}) + f(\{c, d\}) > \overline{\chi}(G, f) = f(\{a, b, c\}) + f(\{d\}) = 12/8$.

**Proof** Directly from Proposition 2.2 □

Given $i < z < j \in X$, let $K_{i,j}^z$ be the set of intervals of $\mathcal{I}[i, j]$ containing point $z$. Notice that $K_{i,j}^z$ is a clique for all $i \leq z \leq j \in X$.

**Lemma 5.3** *For arbitrary fixed $i < j$ in $X$, the following recursion holds:*

$$(12) \quad F(i, j) = \min_{z \in [i,j]} \{f(K_{i,j}^z) + (F(i, z - 1) + F(z + 1, j))\}.$$

**Proof** By Lemma 5.2, there is an optimal partition of $G[\mathcal{I}(i, j)]$ in which a batch is a maximal clique $B^*$. All maximal cliques of $G[\mathcal{I}(i, j)]$ are browsed while considering the minimum in (12). Hence $B^* = K_{i,j}^{z^*}$ for some $z^*$. Given such point $z^*$, every interval in $\mathcal{I}[i, z^* - 1]$ has its terminal endpoint before the initial endpoint of every interval in $\mathcal{I}[z^* + 1, j]$. Hence, the graph $G(\mathcal{I}[i, j] \backslash B^*)$ decomposes into two disconnected subgraphs: $G(\mathcal{I}[x_i, z^* - 1])$ and $G(\mathcal{I}[z^* + 1, j])$. One can therefore solve the problems on these two subgraphs independently. □

The Dynamic Programming algorithm starts from the initial conditions

$$F(i, i) = f(\mathcal{I}[i, i]) \qquad \text{for all } i = 1, \ldots, q.$$

Applying the recursion (12) with increasing subproblem width $x_j - x_i$, it computes an optimal schedule

$$S(x_i, x_j) = \begin{cases} \emptyset & \text{if } \mathcal{I}[i, j] = \emptyset\,; \\ S(i, z^* - 1) \cup B^* \cup S(z^* + 1, j) & \text{otherwise.} \end{cases}$$

The optimum value is $\overline{\chi}(G, f) = F(1, q)$, and $S(1, q)$ is an optimal solution. Since there are $O(q^2) = O(n^2)$ subproblems and $O(q) = O(n)$ candidate values for $z$ in each subproblem, the resulting Dynamic Programming algorithm solves the problem in $O(n^3)$ time. This completes the proof of Theorem 5.1. □

Theorem 5.1 and the associated algorithm can be extended in the following way. A graph $G = (V, E)$ is a ***circular arc graph*** [13] if there exists a set $\{\phi(v) \mid v \in V\}$ of closed arcs of the unit circle, such that two vertices $u$ and $v$ are adjacent in $G$ if and only if the two corresponding arcs $\phi(u)$ and $\phi(v)$ have nonempty intersection. Call a clique $K$ of $G$ a ***Helly clique*** if $\cap_{v\in K}\phi(v)$ is nonempty.

**Corollary 5.4** *For any circular arc graph $G$, and any value-polymatroidal function $f$ on $V(G)$ given by a value oracle, we can compute an optimum partition into Helly cliques in time $O(n^3)$.*

**Proof** Let $X$ be the set of endpoints of the arcs $\phi(v)$, (as in Theorem 5.1). For $i, j \in X$, let $\mathcal{I}[i, j]$ be the set of arcs contained in the portion of the circle in clockwise order between $i$ and $j$. Note that after removing any maximal Helly clique, the remaining arcs are contained in some set $\mathcal{I}[i, j]$. Compute all $O(n^2)$ values as in Theorem 5.1. Compute the best maximal Helly clique afterwards. □

On the other hand, we have the following negative result:

**Theorem 5.5** *[2] [PCliqW] is NP-hard even if $G$ is an interval graphs and $f$ is a polymatroid cost (even if $f$ is given by a rooted-TSP on a tree).*

**Rooted-TSP on trees** Let $T = (W, A)$ be a tree, $l : A \to \mathbb{N}$ and $r \in W$ be the root of $T$. For $U \subseteq W$, let $A(U)$ be the set of arcs spanning $U + r$ and $f(U) := 2\sum_{a\in A(U)} l(a)$. The function $f$ is called a rooted-TSP cost since it is the cost of visiting all nodes in $U \subseteq V$, moving along edges of $A$, starting and finishing the tour from node $r$ (see Figure 3). Such a cost function can easily be shown to be polymatroidal[1]. Complementing Theorem 5.5, [2] gave a 2-approximation for [PCliqW] when $G$ is an interval graphs and $f$ is rooted-TSP on a tree. This has applications in vehicle routing problems with time windows (where the length $l(a)$ represents a travel cost and we assume that the traveling times are negligible compared to the size of the time windows [9]).

[1] In fact, several characterizations of the graphs for which rooted TSP costs are polymatroidal for all edge length can be found in [15]. Based on [15], Jost [16] characterized these graphs as the graphs without $K_{2,3}$ minors.

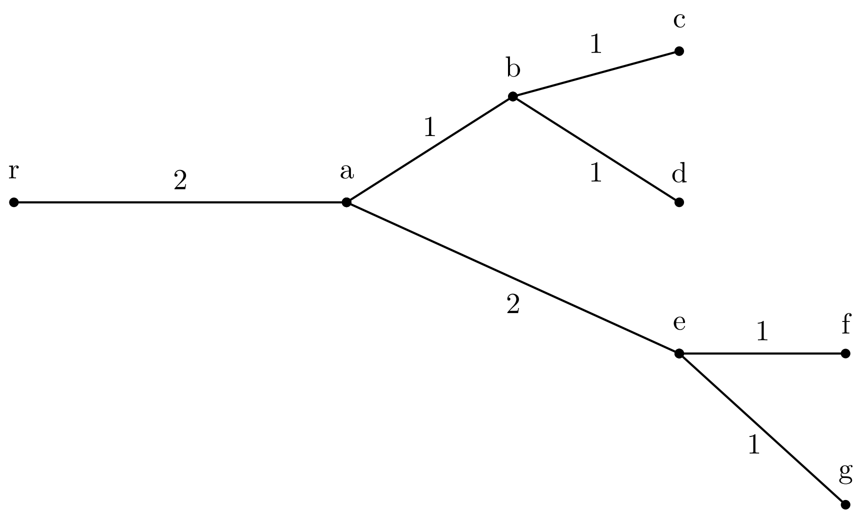


Figure 3: A rooted tree with a length function $l : A \to \mathbb{R}$. The cost associated with a subset $U \subseteq V$ is the length of the arcs spanning $U + r$. For example $f(\{a\}) = 4$, $f(\{a, b, f\}) = 12$ and $f(\{c, d, e, f\}) = 16$.

# 6 Partition into cliques in split graphs

One may wonder if Proposition 2.2 could be applied in more general graphs than interval graphs. A property of interval graphs which is used to prove polynomiality in Theorem 5.1 is that they have a polynomial number of maximal cliques. In this section, we illustrate that this property is not sufficient to ensure polytime solvability of [PCliqW] restricted to value-polymatroidal costs.

A graph $G = (V, E)$ is a ***split graph*** if $V$ can be partitioned into two sets $S$ and $K$ such that $S$ is a stable set and $K$ is a clique. Notice that split graphs have a polynomial number of maximal cliques (at most $|S| + 1$). However, [max-coloring] and [probabilist coloring] are (strongly) NP-hard in split graphs ([3, 8] and [7] respectively). Since the class of split graphs is self-complementary, [PCliqW] is also NP-hard if we restrict to maximum or probabilist cost functions. Moreover, Yannakakis and Gavril [18] proved that the maximum $q$-chromatic subgraph problem is NP-hard for split graph. Unsurprisingly then, Greene-Kleitman's relation doesn't hold for split graphs [5]. However, the "dual problem", that is [PCliqW] with $f(U) := \min\{q, |U|\}$ is trivial. If $q = 1$ this is equivalent to find a partition of $G$ into a minimum number of cliques. If $q \geq 2$, we may assume $\omega(G) = |K|$ (in general, the bipartition $(S, K)$ of a split graph is not unique). Then the partition consisting of all elements of $S$ alone and all vertices of $K$ together in a unique class is optimal. This fact however, does not extend to size-defined cost functions.

**Theorem 6.1** *[PCliqW] is strongly NP-hard even if we restrict $G$ to be a split graph and $f$ to be size-defined and value-polymatroidal.*

**Proof** We reduce the NP-complete problem [X3C] to [PCliqW].

**[X3C] Exact three-set cover**
**INPUT :** A finite set $X$ of size $3m$ and a set $T$ of triples of $X$.
**OUTPUT :** Does there exists a partition of $X$ into $m$ elements of $T$?

Given an instance of [X3C], build the split graph $G = ((T, X), E)$ where $G[T]$ is a stable set and $G[X]$ a clique and $(t, x) \in E$ iff $x \in t$. Let $\psi(0) := 0$, $\psi(1) := \alpha = m+1$ and $\psi(i) := \beta = m+2$ for all $i \geq 2$. We claim that there is a partition of cost not exceeding $m\beta + (|T| - m)\alpha$ if and only if $X$ has a partition into triples of $T$. A partition into triples yields such a cost. Now, assume that $X$ has no partition into triples. Since $T$ induces a stable set, any partition of $V(G)$ into cliques contains at least $|T|$ classes. Those partitions which consist in exactly $|T|$ cliques, are of cost at least $(m + 1)\beta + (|T| - (m + 1))\alpha > m\beta + (|T| - m)\alpha$. Those consisting in at least $|T| + 1$ cliques are of cost at least $(|T| + 1)\alpha > m\beta + (|T| - m)\alpha$. □

# 7 ILP formulation and min-max formula for [PCliqW]

Seen as a partition problem, [PCliqW] can be formulated as an integer linear program, with variables $y$ in $\mathbb{R}^{\mathcal{C}(G)}$ (where $\mathcal{C}(G)$ is the set of cliques of $G$):

(13) (i) $\min f^T y$

(ii) $\displaystyle\sum_{C \ni v} y_C = 1$ for all $v \in V$

(iii) $y_C \in \{0, 1\}$ for all $C \in \mathcal{C}(G)$

Clearly, if $f$ is non-negative, there is no advantage in taking $y_C > 1$. Therefore, $y_C \in \{0, 1\}$ can be replaced by $y_C \geq 0$ and $y_C \in \mathbb{Z}$. Also, if $f$ is non-decreasing, (13) (ii) can be replaced by $\sum_{C \ni v} y_C \geq 1$ (if $y_A = y_B = 1$, $A, B \in \mathcal{C}(G)$ and $A \cap B \neq \emptyset$ then $B \setminus A$ is still a clique of $G$ and we can reset $y_B := 0$ and $y_{B \setminus A} := 1$).

If $f$ is non-negative and non-decreasing, the dual of the linear relaxation of (13) can therefore be written as maximizing $\mathbf{1}^T x$ subject to[2]:

(14) (i) $\displaystyle\sum_{v \in C} x_v \leq f(C)$ for all $C \in \mathcal{C}(G)$

(ii) $x_v \geq 0$ for all $v \in V(G)$

If $G$ is perfect and $f \equiv 1$, (14) is TDI. Also if $G$ is complete and $f$ is submodular, (14) is box-TDI. So in both cases, (14) yields a min-max formula for [PCliqW].

[2]An interpretation of system (14) within the framework of cooperative game theory with cooperation restricted to the cliques of a graph is described in [16].

But there are other famous cases where (14) yields a min-max formula. Greene-Kleitman's theorems can be restated in the following terms: if $G$ is a comparability graph or the complement of such a graph and if $f$ is the rank function of a uniform matroid, system (14) is TDI. Alternatively, Greene-Kleitman's theorems can stated as the box-TDIness of (14) if $G$ is (co)-comparability and $f \equiv 1$ [5]. Note that cliques of the line-graph of a bipartite graph $G$ correpond to subsets of $\delta(v)$ (the set of edges incident with $v$), for some $v \in V(G)$. Now, a common generalization of the polymatroid intersection theorem, of Dilworth's truncation and of min-max relations for bipartite $b$-matching can be stated as box-TDIness of (14) if $G$ is the line-graph of a bipartite multigraph and $f$ is submodular. More precisely we have (see section 48.3 of [17] for an idea of the proof and Chapter 60 for extensions),

**Theorem 7.1 (Submodular bipartite matchings polyhedron)** *[16]*
*Let $G = ((A, B), E)$ be a bipartite multi-graph and for all $v \in A \cup B$ let $f_v$ be a submodular function on $\delta(v)$, then the following system is box-TDI*

$$(15) \quad \sum_{e \in F} x_e \leq f_v(F) \text{ for all } v \in A \cup B \text{ and } \emptyset \neq F \subseteq \delta(v)$$

In view of these results, it seems reasonable to expect system (14) to provide other min-max relations for [PCliqW]. However, the linear relaxation of (13) does not always have an integral optimal solution, even if $G$ is an interval graph and $f$ is a value-polymatroidal set function as shown in Figure 4 (other examples for which $G$ is perfect, $f$ is a submodular but the linear relaxation of (13) has no integral optimal solution are provided in [16]).

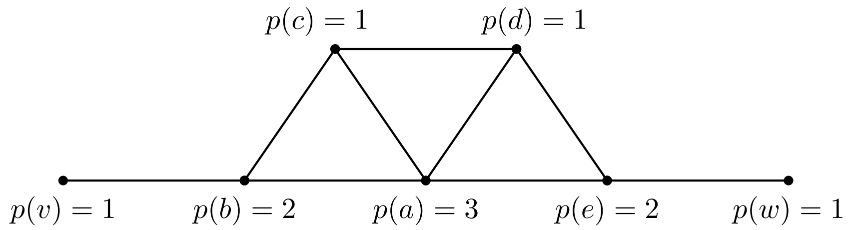


Figure 4: Let $f$ be the max-batch cost defined by $p$. An optimal solution to the linear relaxation of (13) is given by $y_C = 1/2$ if $C \in \{\{v\},\{b,v\},\{a,b,c\},\{a,d,e\},\{c,d\},\{e,w\},\{w\}\}$ and $y_C = 0$ otherwise. The cost of this fractional partition is 13/2. Optimality can be checked using an $x$ maximizing $\mathbf{1}^T x$ subject to (14), for instance $x(a) := 3/2$, $x(c) = x(d) := 1/2$ and $x(b) = x(e) = x(v) = x(w) := 1$.

## 8 Conclusion and extension

Although we were able to compute an optimum solution for [PCliqW] when $G$ is an interval graph and $f$ is value-polymatroidal, we were unable to complement this result by a min-max formula. This issue could be linked with the following extension: consider the problem of multi-partition into cliques, that is, generalize the ILP (13) by replacing constraints (ii) by $\sum_{C \ni v} y_C = d_v$, where $d_v \in \mathbb{N}$ is the covering demand associated to vertex $v$. The complexity of this problem is left open and, to the best of our knowledge, is beyond the scope of our dynamic program. A polytime algorithm for this last problem might shed new light on the structure of interval graphs and therefore be useful to solve various problems on interval graphs.

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